V. Morbidity, Impairment, and Mortality

Editor's Note
In this chapter, we are able to present only a sampling of the very large multi-disciplinary field of research concerning morbidity and impairment as precursors of death. Underlying all of it is the phenomenon of heterogeneity, since individuals in a population are not homogeneous. The characteristics of certain individuals affect the risk of dying at ages that could be considered as premature. The next paper by Vaupel and others uses life table methods to explore the impact of heterogeneity in frailty on the dynamics of total mortality.

The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality
James W. Vaupel, Kenneth G. Manton, and Eric Stallard

Demography 16, 3 (1979), pp. 439-54

INTRODUCTION
That individuals differ substantially in their endowment for longevity is well known (e.g., Strehler, 1977; Keyfitz, 1978), yet currently used life table methods ignore this heterogeneity. A handful of recent papers (e.g., Shepard and Zeckhauser, 1975, 1977; Tolley et al., 1978; Manton and Stallard, 1979) have focused on how much of the variation in individual susceptibility to specific causes of death. In this paper, life table methods are developed for populations whose members differ in their general susceptibility to all causes of death. The methods are then used to explore the impact of such heterogeneity in frailty on the dynamics of total mortality. The model yields insights into intriguing and potentially important implications:

1. Mortality rates for individuals may increase faster with age than observed mortality rates for cohorts.
2. The life expectancy of those whose lives might be saved by some health or safety intervention may be less than currently estimated.
3. Past progress against mortality may be underestimated, and as a consequence, predictions of future progress against mortality may be too low.
4. Current methods of computing life tables may confound past mortality experiences with current ones. Indeed, if current mortality rates had been unchanged, mortality rates presented in life tables, as currently calculated, might increase in the future.
5. Heterogeneity in frailty may be a factor in observed declines and reversals with age of mortality differentials between pairs of populations.

This paper is organized into four sections. The first contains a model of individual differences in frailty. Frailty is defined and then assumptions are made concerning the distribution of frailty in populations.

The second section of the paper explains how mortality rates for individuals at any specified level of frailty can be estimated on the basis of the mortality rates calculated for the cohort to which the individuals belong. Next, as an illustration, individual and cohort mortality rates are compared using data for Swedish females born in 1875. It is argued that it is more informative to measure historical progress in reducing mortality on the basis of changes in the force of mortality for individuals at specific levels of frailty rather than changes in cohort mortality rates. Swedish mortality data are again used to illustrate the empirical differences between these two kinds of measures.

The third section of the paper focuses on period life tables. If heterogeneity in frailty is substantial, standard life table methods will not correctly represent the current pattern of mortality. We propose a method for calculating adjusted period life tables that do reflect current mortality, given population heterogeneity. The standard and adjusted period life tables are compared for Swedish females in 1975.

In the final section of the paper, we extend the methodology to analyze mortality differentials between two populations. To illustrate the effect of population heterogeneity, we contrast population versus individual mortality rates for Swedish females and males in 1975.

A MODEL OF INDIVIDUAL DIFFERENCES IN FRAILTY

The Definition of Frailty
Let \( \mu(x, y, z) \) be the force of mortality for an individual in population \( f \), at exact age \( x \), at some instant in time \( y \), and with a "frailty" of \( z \). Demographers have traditionally studied how mortality rates vary across populations, by age, and over time; what is unusual about this formulation is the inclusion of a fourth variable \( z \) to allow for individual differences in mortality rates. This "frailty" variable could be defined in numerous ways; we have chosen to define it in terms of the following relationship:

\[
\mu(x, y, z) = \mu_0(x, y, z') z' = z, \quad (1a)
\]

or, alternatively,

\[
\mu(x, y, z) = z \cdot \mu_0(x, y, 1). \quad (1b)
\]

An individual with a frailty of 1 might be called a "standard" individual. Then, an individual with a frailty of 2 is twice as likely to die, at any particular age and time, as the standard individual; an individual with a frailty of 1/2, on the other hand, is only one-half as likely to die.

We chose to define frailty in terms of the force of mortality, \( \mu \), rather than the age-specific probability of death, \( q_x \), because of two major difficulties in defining frailty in terms of \( q_x \). First, since \( q_x \) is bounded above by one, the range of \( z \) would necessarily also be bounded above. Second, \( q_x \) is known to be a nonlinear function of the size of the age interval used. Consequently, we will develop the model in terms of \( \mu \), and later give the necessary equations to calculate the \( q_x 's \) for life table construction.

Note that the definition of frailty assumes that each individual is born at a certain level of relative frailty and stays at this level all his or her life. The definition does not imply, however, that individuals at the same level of frailty are identical—even if they are contemporaries from the same population. The variable \( \mu \) merely measures the likelihood of death; the exact moment of death will be determined by various individual differences beyond population group, age, date of birth, and frailty level. Frailty, as used here, is just one component—and a very special age-invariant one—of an individual's complex makeup.

The assumption that frailty is constant for individuals is a first approximation. Recognizing, in life table computations, the known heterogeneity in populations.

A more complete model would recognize (a) that frailty is probably a result of a large number of factors, (b) that frailty is probably not constant for life, (c) that frailty should probably include differential susceptibility to cause-specific mortality, and (d) that mortality due to chronic
Thus if a standard individual has a 50 percent chance of surviving to some age, an individual with a frailty of 2 will only have a 25 percent chance of surviving to this age and an individual with a frailty of 3 only a 12.5 percent chance.

The Distribution of Frailty

Let \( \mu_i(x, y) \) be the force of mortality for a cohort of individuals from population group \( i \) at age \( x \) at time \( y \). That is, let

\[
\mu_i(x, y) = \int_0^\infty \mu(x, y, z) \cdot f(z) dz.
\]

where \( f(z) \) is the p.d.f. (probability density function) of frailty at age \( x \) among the surviving individuals in the cohort. That the force of mortality for the cohort is indeed the same as the average force of mortality for the surviving individuals in the cohort is shown in the fourth section of the Appendix.) Average frailty in the cohort, \( \bar{z} \), is defined by

\[
\bar{z}(x, y) = \int_0^\infty z \cdot f(z) dz.
\]

Consequently, it follows from the definition of frailty in equation (1b) that

\[
\bar{z}(x, y) = \mu(x, y, 1) \cdot \bar{z}(x, y),
\]

or, in simpler notation,

\[
\bar{z} = \mu \cdot \bar{z}.
\]

Therefore, independent frailty \( x \) with high values of \( x \) will tend to die first. Thus, \( \bar{z} \), the average frailty of the surviving cohort, will decline with age. Consequently, equation (6b) implies that the force of mortality for individuals increases more rapidly than for the cohort the individuals belong to: in this sense, individuals "age faster" than cohorts. An important implication is that studies of human aging based on cohort mortality data may be systematically biased or based on erroneous functional forms.

The precise nature of the relationship between individual and cohort aging depends on the distribution of frailty among individuals. This paper assumes that frailty at birth is gamma distributed with

\[
\Gamma(k) = \lambda^k \cdot \lambda^{-k-1} \cdot e^{-\lambda x}/\Gamma(k),
\]

where \( \lambda \) and \( k \) are the parameters of the distribution. The mean and variance of a gamma variate are given by

\[
\bar{z} = k/\lambda
\]

and

\[
\sigma^2 = k/\lambda^2.
\]

Figure 1 plots the shape of gamma p.d.f.'s for three values of \( k \) that will be used in the empirical sections of this study: \( k = 1, 4, \) and 8. These three values were selected because they represent a broad range of distributions of frailty. A value of \( k \) of 1 may at first seem extreme, but some empirical research on mortality crossovers (Manton et al., 1979) suggests values of \( k \) around 1 and some empirical work with certain diseases, e.g., lung cancer (Manton and Stallard, 1979) suggests values of \( k \) much less than 1. In each case, the mean \( \bar{z} \) is set equal to 1, so that \( \lambda = k \) and \( \sigma^2 = 1/k \).

The gamma distribution was chosen because it is analytically tractable and readily computable. It is a flexible distribution that takes on a variety of shapes as \( k \) varies: when \( k = 1 \), it is identical to the well-known exponential distribution; when \( k \) is large, it assumes a bell-shaped form reminiscent of a normal distribution. Frailty cannot be negative and the gamma distribution is, along with the log-normal and Weibull distribution, one of the most commonly used distributions to model variables that are necessarily positive. At least one other study of heterogeneity (Shephard and Zeckhauser, 1977) also uses the gamma distribution for these various reasons.

It turns out, as shown in the first section of the Appendix, that the assumption that frailty at birth is gamma distributed yields some useful mathematical results, including:

1. Frailty among the survivors at any age \( x \) is gamma distributed with the same value of the shape parameter \( k \) as at birth. The value of second parameter, however.

\[
\lambda(x) = \lambda + H(x).
\]

The mean frailty of the survivors is therefore given by

\[
\bar{z}(x) = \bar{z} \cdot k/(k + \lambda + H(x)),
\]

where \( \bar{z} \) is the mean frailty of the cohort at birth (as given in equation 8a). When \( k = 1 \) and \( \bar{z} \), which is essentially an arbitrary scaling value, is set equal to 1, equation (9b) reduces to

\[
\bar{z}(x) = 1/(1 + H(x)).
\]

This equation clearly illustrates how the average frailty of a cohort decreases as the cumulative hazard suffered by the cohort increases.

2. Frailty among those who die at any age \( x \) is also gamma distributed, with the same parameter \( \lambda(x) \) as among those surviving to age \( x \) but with shape parameter \( k + 1 \). This implies that the mean frailty of those who die, \( \bar{z}' \), is somewhat greater than the mean frailty of the survivors:

\[
\bar{z}'(x) = \bar{z}(x) \cdot (k + 1)/k.
\]

This result may prove useful in refining calculations of the benefits of programs to "save" lives (or, more precisely, to delay deaths).

Computing Individual Life Tables from Cohort Life Tables

If frailty is gamma distributed, a simple formula (derived in the second section of the Appendix) relates the force of mortality for an individual at any age \( x \) and any level of frailty \( z \) to the cohort force of mortality:

\[
\mu(x, z) = \mu(x) \cdot (z/\bar{z}(0)) \cdot (\bar{z}(x))^{-1/k}
\]

where \( \mu(0) \) is the mean frailty of the cohort at birth and \( \bar{z}(0) \) is the proportion of the cohort surviving at age \( x \). For theoretical purposes equation (11) defines the relationship between individual and cohort mortality, but it is inconvenient for empirical calculations since mortality rates are published in terms of cohort age-specific frailty.
to compute \( q_z(z) \), the age-specific probability of death for an individual of frailty \( z \), from data on cohort survival, \( \delta_z \):

\[
q_z(z) = 1 - \exp\left(-k \cdot \left(\frac{z}{\delta_z}\right)\right) \\
\cdot \left(\delta(z+n) - \delta(z)\right). \tag{12}
\]

And \( \delta(x) \) can be calculated at any exact age \( x \), from cohort age-specific mortality rates:

\[
\delta(0) = 1,
\]

and

\[
\delta(x) = \prod_{t=1}^{x-1} (1-\delta_t) \quad x \geq 1. \tag{13}
\]

The parameter \( \delta(0) \), which measures the mean frailty of the cohort at birth, is essentially just a scaling factor that, for most purposes, can simply be set equal to one. That leaves a single parameter \( k \), which measures the degree of heterogeneity in frailty: the greater \( k \), the less heterogeneity. Given an estimate of \( k \), equations (12) and (13) allow for the translation from published cohort life tables to life tables for individuals at any specific level of frailty.

 Cohort vs. Individual Mortality for Swedish Females

To illustrate the nature of the difference between cohort and individual mortality, it is useful to look at some empirical results. We decided to base the empirical calculations in this paper on data for Swedish females and males because high quality mortality data are available for nearly two centuries for these populations. Since the results presented here are intended to be illustrative rather than definitive, we have relegated discussion of the sources of these data and various interpolations and calculations performed on them to a separately available working paper (Vaupel et al., 1979).

Figure 2 compares cohort mortality rates with mortality rates for individuals of standard frailty (i.e., \( z = 1 \)), at three values of \( k \): \( k = 1, 4, \) and 8. The mortality

1875.

Figure 1 plotted the gamma p.d.f. for these three values of \( k \). When \( k \) is infinite, there is no variability in frailty and cohort and individual mortality rates are identical.

Note in Figure 2 that as \( k \) increases (i.e., as variability in frailty decreases), mortality rates for standard individuals become more like the observed cohort rates. Also, note that the effects of selection increase with age. The most striking feature of these plots is the rapid increase in mortality rates when \( k = 1, 4, \) or even 8. This implies that \( k \) heterogeneity in frailty is substantial, the maximum life span of an individual of a given frailty is well determined within a few years.

Figure 3 compares cohort mortality rates, \( \tilde{q}_z \), with individual mortality rates, \( q_x \), for individuals at four levels of relative frailty: \( z = 1/4, 1/2, 1, \) and 2. In calculating each of the four individual mortality curves, \( k \) was assumed to equal 1. The curves, as before, are based on estimates of the mortality experience of Swedish females born in 1875.

The four individual mortality curves plotted in Figure 3 clearly illustrate the effect of relative frailty on individual mortality. The manner in which the cohort mortality curve cuts through the individual curves at lower and lower values of frailty demonstrates the fact that as death selectively removes the relatively frail, the average frailty of a cohort decreases. Cohort mortality rates thus increase less rapidly than mortality rates for any individual in the cohort.

 How Much Progress Has Been Made in Reducing Mortality?

Over the last century or two, mortality rates at most ages have declined at the same time that the proportion of cohorts that reach any particular age has increased. Customary measures of progress consider only changes in cohort mortality—these measures ignore increases in survivorship. Consequently, to the extent that heterogeneity in frailty is significant,
Figure 3: Mortality rates for individuals of different frailty (z = 2, 1, 1/2, 1/4) from frailty distribution with K=1.

KEY
- □ OX, Z = 2
- ◇ OX, Z = 1
- ◯ OX, Z = 1/2
- ▲ OX, Z = 1/4
- ○ OX, K = COHORT MORTALITY RATE

Figure 2: Mortality rates for standard individual (z=1) for K=1,4,8,∞.

KEY
- Δ OX, K = 1
- ○ OX, K = 4
- △ OX, K = 8
- × OX, K = INFINITY
The table below shows the period life tables for different ages and years:

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</table>
\[ \mu(x,y,z) = \mu(x,y^*,z) \text{ all } x, \]
\[ \text{all } y, \text{ all } y > y^*, \]
where \( y^* \) is the point in time when progress ceases. In such a steady-state situation, the future values of cohort mortality rates (as measured by \( \mu \) of \( \tilde{q} \)) will, surprisingly enough, increase to levels greater than their initial levels.

To see this, let \( \tilde{\mu} \) and \( \tilde{\tilde{\mu}} \) represent the values of \( \mu \) and \( \tilde{q} \) in the long run. It follows from equation (11) that
\[ \tilde{\mu} = \mu \cdot (\tilde{y}(0) \cdot \tilde{z}(0))^{1/2}. \] (14)

Solving equations (11) and (14) for \( \mu \) yields, respectively,
\[ \mu = \tilde{\mu} / (\tilde{z}(0) \cdot \tilde{y}(0)) \]
and
\[ \mu = \tilde{\mu} / (\tilde{z}(0) \cdot \tilde{y}(0)). \] (15)

Equating the right-hand side of these equations and solving for \( \mu \) then yields:
\[ \mu = \tilde{\mu} / \tilde{z}(0) \cdot \tilde{y}(0). \]

This equation makes apparent the relationship between \( \tilde{\mu} \), the currently observed cohort force of mortality, and \( \tilde{\mu} \), the cohort force of mortality at current mortality rates, i.e., the cohort force of mortality that would be eventually observed if current levels of mortality remained unchanged. In particular, to the extent progress has been achieved in the past in reducing mortality rates, a greater proportion of individuals will survive to any particular age than before, i.e., \( \tilde{\mu} \) will be greater than \( \tilde{\mu} \). As a result, future populations at any age will tend to be larger on average than current populations. The equation indicates that unless future progress in reducing mortality rates is sufficient to counterbalance this effect, future mortality rates will rise—even if some progress is actually being made. As in Lewis Carroll's Through the Looking Glass, it may take "all the running you can do to keep in the same place."

**Adjusted Period Life Tables**

Period life tables are designed to represent current patterns of population mortality. Standard life table methods construct period life tables according to the same basic set of equations as cohort life tables. In particular, period (and cohort) life tables are based on the cohort age-specific probabilities of death, the \( \tilde{q} \)'s. As indicated above, however, the frailty hypothesis suggests that the values of \( \tilde{q} \)—and hence the values of \( \tilde{\mu} \), as well—are really a mixture of current and past mortality experiences. Thus the values of \( \tilde{q} \), not \( \tilde{\mu} \), are the correct mortality rates for a period life table, i.e., the mortality rates that a newborn cohort would experience as it aged if current patterns of mortality remained unchanged.

As shown in the Appendix, it is possible to compute \( \tilde{\mu} \) on the basis of the following formula:
\[ \tilde{\mu}_x = 1 - (1 + (\tilde{z}(x)/\tilde{y}(x)))^{1/2} \cdot \left[(1/(1 - \tilde{\mu}_x)) - 1]\right]^{1/2}. \] (16)
The values of \( \tilde{\mu}_x \) can be obtained from standard period or cohort life tables and values of cohort survivorship, \( \tilde{z}(x) \), at any exact age \( x \), can be calculated from cohort life tables as indicated in equation (13). The values of \( \tilde{\mu}_x \), which represent period survivorship and thus are analogous to the values often designated by \( \mu_x \) or \( \tilde{z}(x) \), can be iteratively calculated as follows:
\[ \tilde{z}(0) = 1, \]
and
\[ \tilde{z}(x) = \tilde{z}(x - 1) \cdot (1 - \tilde{\mu}_{x-1}), \ x \geq 1. \] (17)

Table 2 illustrates the difference it would make to base period life tables on \( \tilde{z} \) rather than \( \tilde{\mu} \). The table presents four alternative estimates of the life expectancy of Swedish females in 1975, as currently calculated (under the implicit assumption that \( k = 1 \)) and as adjusted for \( k = 1 \) and \( k = 4 \). Note that the statistics in this table pertain not to the standard individual but to the entire population. The statistics indicate that life expectancy at current mortality rates may be overestimated by customary life table calculations and that this overestimation becomes more significant as heterogeneity increases.

**Table 2: Population Life Expectancy for Swedish Females at 1975 Mortality Rates, Under Various Assumptions About Heterogeneity**

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<th>Adjusted: ( k = 4 )</th>
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**COMPARISON OF TWO HETEROGENEOUS POPULATIONS**

In addition to being useful in understanding patterns of mortality within a single population, the hypothesis of heterogeneity in frailty may explain some puzzling anomalies in period mortality differentials between populations. Variation in the force of mortality in youth or middle age is much more substantial across countries and various population groups than variation among the elderly. For most pairs of populations—e.g., for U.S. whites vs. blacks, for U.S. males vs. females, or for Americans vs. Swedes—mortality differentials tend to decline and even reverse with age (see Thornton and Nam, 1972; Nam and Okay, 1977; Manton et al., 1979; Strichler, 1977). One striking and surprising reversal concerns Puerto Rico which, among countries for which good mortality statistics are available, is the world's leader in life expectancy at age 65 (Vaupel, 1976). Such convergence and cross-over of mortality differentials might be at least partially caused by decreases in the average frailty of a population cohort at later ages as frailter members are removed by mortality.

To see this, consider two cohorts for which the cohort force of mortality is described by a series of values \( \mu_1 \) and \( \mu_2 \) and the force of mortality for individuals in the cohort by a series of values \( \mu_1 \) and \( \mu_2 \). Equation (11) implies that at any age and point in time:
\[ \tilde{\mu}_x / \mu_x = (\tilde{\mu}_x / \mu_2) \cdot (\tilde{z}_x / \tilde{z}_2). \] (18a)

In the special case where \( \tilde{z}_2(0) = \tilde{z}_2(0) \) and \( k_1 = k_2 \), equation (18a) reduces to:
\[ \tilde{\mu}_x / \mu_x = (\tilde{\mu}_x / \mu_2) \cdot (\tilde{z}_x / \tilde{z}_2). \] (18b)

As shown in the second section of the Appendix,
\[ \tilde{z}(x) = (1/(1+\tilde{\mu}_x)) \cdot \tilde{z}(x+1)/\tilde{z}(x). \] (18c)
Consequently, assuming \( \hat{z}(0) = 1 \), equation (18b) can be rewritten as:

\[
\hat{\mu} / \mu_1 = (\mu_2 / \mu_1) \cdot [(k + H_3)/(k + H_2)].
\]  

(20)

The second cohort might be called "disadvantaged" relative to the first if \( \mu_2 > \mu_1 \) and if, as a result of previous excess mortality, \( H_3 > H_2 \). Consider the special case where \( \mu_2 / \mu_1 \) equals \( H_3 / H_2 \); that is, suppose the current level of disability is the same as the overall historical level. In this case, equation (20) implies that \( H_3 > H_2 \) increase with age, \( \hat{\mu} \) will approach \( \mu_2 \); i.e., the cohort mortality rates will converge. There will not, however, be any cross-over; \( \hat{\mu} \) will never fall below \( \mu_1 \). For a cross-over to occur the current mortality differential for individuals must be somewhat less than the overall differential in the past; the precise condition is

\[
\mu_2 / \mu_1 < (k + H_3)/(k + H_2).
\]

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A simple example illustrates these dynamics. Suppose that \( H_3 = H_2 = 2 \), so that historically the disadvantaged cohort has suffered twice the force of mortality of the advantaged cohort. Suppose that \( H_1 = 1 \). And suppose that \( k = 1 \). In this case, equation (19) indicates that one-half of the advantaged cohort and one-third of the disadvantaged cohort are alive. Thus, as can be seen from either equation (18b) or equation (20), the cohort mortality differential, \( \hat{\mu} / \mu_1 \), will be two-thirds of the individual mortality differential, \( \mu_2 / \mu_1 \). Consequently, if \( \mu_2 / \mu_1 \) like \( H_3 / H_2 \) equals 2, the cohort differential \( \hat{\mu} / \mu_1 \) will be 1.33. If, however, \( \mu_2 / \mu_1 \) equals 1.2, \( \hat{\mu} / \mu_1 \) will equal 0.8. Although individuals at any specific level of frailty in the disadvantaged cohort suffer a 20 percent higher force of mortality than individuals in the advantaged cohort, the disadvantaged cohort will appear to be doing 20 percent better than the advantaged cohort.

Table 3 displays population and individual mortality rates for Swedish males and females in 1975. In calculating the individual rates, it was assumed that frailty at birth for males and females was identically distributed, with a mean of 1 and a \( k \) of 1. A value of 1.0 was selected for \( k \) because it produced a relatively constant male/female ratio after age 30, i.e., it eliminated the convergence in the unadjusted cohort data. The value of 1.0 was the largest value of \( k \) that eliminated the convergence; smaller values yielded divergence. As in Table 1 and for essentially the same reasons, the male and female cohort and individual mortality rates are measured in terms of \( q_5 \) but the comparisons of the rates are in terms of the ratio of \( q_m / q_f \) to \( \mu_5 / \mu_f \).

**DISCUSSION**

The three variables usually considered in studies of mortality—population group, age, and year—are well defined and readily measurable. The variable "frailty," on the other hand, could be defined in any of a number of ways and, however defined, is difficult to measure. As a consequence, demographers have largely ignored heterogeneity in frailty, presumably in the hope that such neglect would result in estimation errors that are small and centered around zero.

The results of this study, however, suggest that ignoring frailty may result in biased estimates. Individual aging rates, past and future progress in reducing mortality, and mortality differentials between populations may be underestimated. On the other hand, current life expectancy and potential gains in life expectancy from averting specific causes of death may be overestimated. Furthermore, illustrative calculations based on Swedish mortality data suggest the possible magnitude of bias. Given the importance of understanding the dynamics of mortality in demographic and biomedical research and in public policy analysis, the results indicate that heterogeneity in frailty is an area of research that may well prove worthy of considerable attention.

**APPENDIX**

1. In this study we have defined frailty, \( z \), as a continuous random variable. In mortality analysis age of death, \( a \), may also be considered to be a continuous random variable. This section of the Appendix derives the joint probability density function (p.d.f.) of \( a \) and \( z \) and various marginal and conditional p.d.f.'s involving \( a \) and \( z \).

As indicated in the text, we assume in this study that the marginal p.d.f. of \( z \), \( f_z(z) \), is a gamma p.d.f.: 

\[
f_z(z) = \lambda^z z^{z-1} e^{-\lambda z}/(\Gamma(k)).
\]  

(1.1)

The force of mortality by definition is given by:

\[
\mu(a,z) = f_{a|z}(a|z)/f_z(z),
\]  

(2.2)

where \( f_{a|z}(a|z) \) is the p.d.f. of a conditional on \( z \). Solving this equation for \( f_{a|z}(a|z) \) and substituting equations (1c), (3), and (4) yields:

\[
f_{a|z}(a|z) = z \cdot \mu(a) \cdot e^{-z \cdot \lambda(a)}/(\Gamma(k + 1)).
\]  

(2.3)

This p.d.f. is clearly a gamma p.d.f. with parameters \( \lambda(a) \) and \( k + 1 \).

Because survival at age \( a \) implies an age of death greater than \( a \), integrating (2.4) with respect to age of death from \( a \) to \( \infty \), remembering that
\[ \int_1 f_{a \mid s} (a/z)da = e^{-s \cdot H(a)}, \]
and then normalizing the result, yields the p.d.f. of \( z \) in the surviving population at age \( x \):
\[ f_s(z) = (\lambda(x))^{x \cdot z} \cdot e^{-\lambda(x) \cdot x}/(x/k). \quad (A.9) \]
This p.d.f. is also a gamma p.d.f., but with parameters \( \lambda(x) \) and \( k \).

II. To derive the individual force of mortality \( \mu(x) \) as given in equation (11), observe that it follows from (6b) and (8a) that
\[ \dot{\mu}(x) = \mu(x) \cdot k / \lambda(x). \quad (A.10) \]
By definition the cohort force of mortality is given by:
\[ \dot{\mu}(x) = f_s(x)/\dot{s}(x). \quad (A.11) \]
Solving (A.11) for \( \dot{s}(x) \) and then substituting (A.6) and (A.10) yields:
\[ \dot{s}(x) = (\lambda/x \cdot \lambda(x))^x. \quad (A.12) \]
The mean frailty of the cohort at age \( x \) can be derived from the parameters of \( f_s(z) \) in (A.9):
\[ \dot{s}(x) = k / \lambda(x). \quad (A.13) \]
Equations (A.12) and (A.13) together imply that:
\[ \dot{s}(x) = \lambda(0) \cdot (\lambda(x))^{1/\lambda}. \quad (A.14) \]
(since \( \lambda = \lambda(0) \). Solving (6b) for \( \mu(x) \) and substituting the result and (A.14) in (1c) yields the form of \( \mu(x) \) given in (11).

Also note that equations (A.12) and (A.13) and the definition of \( \lambda(x) \) as
\[ \lambda(x) = \lambda + H(x) \]
imply the formula for \( \dot{s} \) as used in equation (19).

III. To derive the individual age-specific probability of death \( s_{a \mid s}(z) \) given in (12), note that by definition:
\[ s_{a \mid s}(z) = s(x, z) \cdot (1 - q_s(z)). \quad (A.15) \]
Using the formulas for \( s \) in (3) and (4) it
\[ \int_s^\infty s_{a \mid s}(z)dz = 1 - \exp(-H(x+n+z) - H(x)). \quad (A.16) \]
As indicated in (2a), integrating (11) with respect to age from 0 to \( x \), remembering
\[ \dot{s}(x) = \exp\left\{ - \int_x^0 \dot{\mu}(t)dt \right\}. \]
yields the cumulative hazard function for individuals:
\[ H(x) = k \cdot (x/\lambda(0)) \cdot (1/\lambda(x))^{1/\lambda}. \quad (A.18) \]
When (A.17) is substituted in (A.16), equation (12) is the result.

To derive the adjusted period mortality rates \( \dot{q}_a \) given in (16) observe that the cumulative hazard function for individuals derived from the adjusted period life table can be expressed, by analogy to (A.17), as:
\[ H(x) = k \cdot (x/\lambda(0)) \cdot (1/\lambda(x))^{1/\lambda}. \quad (A.18) \]
Thus, the individual age-specific probability of death \( s_{a \mid s}(z) \) given in (A.18) may be evaluated using either (A.17) or (A.18). Equating the two forms and using (13) and (17) to introduce the variables \( \dot{q} \) and \( \dot{q} \)
yields equation (16).

IV. Equation (5) assumes that the force of mortality for a cohort equals the average force of mortality for the surviving individuals in the cohort. Though this is an intuitively plausible assertion, its truth is not immediately apparent. Here we indicate the required steps for a formal proof.

The force of mortality is defined by:
\[ \dot{\mu}(x) = f_s(x)/\dot{s}(x). \quad (A.11) \]
The p.d.f. \( f_s(x) \) is given by:
\[ f_s(x) = \int_0^x f_s(a, s, x)dz. \quad (A.19) \]
Equations (A.4) and (A.3) imply that this can be rewritten as:
\[ \int_0^x f_s(a, s, x)dz = \int_0^x f_s(a, x)dz. \quad (A.20) \]
It is apparent that
\[ f_s(x) = f_s(x) \cdot s(x, x)/\dot{s}(x). \quad (A.21) \]
Reexpressing (A.20) in terms of \( f_s(z) \) and then substituting the result in (A.11) yields:
\[ \dot{\mu}(x) = \mu(x) \cdot \int_0^x z \cdot f_s(x)dz. \quad (A.22) \]
The integral in (A.22) simply equals \( \dot{\mu}(x) \). Thus, we have equation (6), which can be true if and only if equation (5) is true.

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